Non-commutative Gauge Theories and Seiberg–Witten Map to All Orders[1]

Kayhan ÜLKER

Feza Gürsey Institute *
Istanbul, Turkey
(savefezagursey.wordpress.com)

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Outline

- Moyal $*$–product.
- Non-commutative Yang–Mills $(\hat{A}\,,\,\hat{\Lambda})$
- Seiberg – Witten Map $(\hat{A} \to \hat{A}(A,\theta)\,,\,\hat{\Lambda} \to \hat{\Lambda}(A,\alpha,\theta))$
- Construction of the maps order by order leads to all order solutions
- All order solutions can be obtained from SW differential equation
- SW map of other fields $(\hat{\Psi} \to \hat{\Psi}(A,\psi,\theta))$
- Conclusion

Moyal $*$-product

- The simplest way to introduce non-commutativity is to use (Moyal) *–product

$$f(x) * g(x) \equiv exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}\right) f(x)g(y)|_{y\to x}$$
$$= f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_{\mu}f(x)\partial_{\nu}g(x) + \cdots$$

  for *real CONSTANT antisymmetric parameter* θ !

- *–commutator of the ordinary coordinates :

$$[x^{\mu}, x^{\nu}]_{*} \equiv x^{\mu} * x^{\nu} - x^{\nu} * x^{\mu} = i\theta^{\mu\nu}.$$

NC QFT models can then be obtained by replacing the ordinary product with the *–product !

NC Yang–Mills Theory

- The action of NC YM theory is written as :

$$\hat{S} = -\frac{1}{4} Tr \int d^4x \hat{F}^{\mu\nu} * \hat{F}_{\mu\nu} = -\frac{1}{4} Tr \int d^4x \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_*$$

is the NC field strength of the NC gauge field $\hat{A}$.

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is the NC field strength of the NC gauge field $\hat{A}$.

- The action is invariant under the NC gauge transformations:

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu = \partial_\mu \hat{\Lambda} - i[\hat{A}_\mu, \hat{\Lambda}]_* \equiv \hat{D}_\mu \hat{\Lambda}$$
$$\hat{\delta}_{\hat{\Lambda}} \hat{F}_{\mu\nu} = i[\hat{\Lambda}, \hat{F}_{\mu\nu}]_*.$$

Here, $\hat{\Lambda}$ is the NC gauge parameter.

Seiberg–Witten Map

One can derive both conventional and NC gauge theories from string theory by using different regularization procedures *(SW'99)*.

Let A_μ and α to be the ordinary counterparts of $\hat{A}_\mu$ and $\hat{\Lambda}$ respectively.

$\Longrightarrow$ There must be a map from a commutative gauge field A to a noncommutative one $\hat{A}$, that arises from the requirement that *gauge invariance should be preserved* !

$$\hat{A}(A) + \hat{\delta}_{\hat{\Lambda}} \hat{A}(A) = \hat{A}(A + \delta_\alpha A)$$

where δ_α is the ordinary gauge transformation :

$$\delta_\alpha A_\mu = \partial_\mu \alpha - i[A_\mu, \alpha] \equiv D_\mu \alpha.$$

This map can be rewritten as

$$\hat{\delta}_{\hat{\Lambda}}\hat{A}_\mu(A;\theta) = \hat{A}_\mu(A+\delta_\alpha A;\theta) - \hat{A}_\mu(A;\theta) = \delta_\alpha \hat{A}_\mu(A;\theta)$$

Since SW map imposes the following functional dependence :

$$\hat{A}_\mu = \hat{A}_\mu(A;\theta) \quad , \quad \hat{\Lambda} = \hat{\Lambda}_\alpha(\alpha, A;\theta).$$

$\Longrightarrow$ one has to solve

$$\hat{\delta}_{\hat{\Lambda}}\hat{A}_\mu(A;\theta) = \delta_\alpha \hat{A}_\mu(A;\theta)$$

simultaneously for $\hat{A}_\mu$ and $\hat{\Lambda}_\alpha$ and it is difficult !

Now, remember the ordinary gauge consistency condition

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{-i[\alpha,\beta]}.$$

(check for instance for $\delta_\alpha \psi = i\alpha\psi$)
Let, Λ be a Lie algebra valued gauge parameter $\Lambda = \Lambda_a T^a$. For the non-commutative case, we get

$$(\delta_{\Lambda_\alpha}\delta_{\Lambda_\beta} - \delta_{\Lambda_\beta}\delta_{\Lambda_\alpha})\hat{\Psi} = \frac{1}{2}[T^a, T^b]\{\Lambda_{\alpha,a}, \Lambda_{\beta,b}\}_* * \hat{\Psi} + \frac{1}{2}\{T^a, T^b\}[\Lambda_{\alpha,a}, \Lambda_{\beta,b}]_* * \hat{\Psi}$$

Only a $U(N)$ gauge theory allows to express $\{T^a, T^b\}$ again in terms of T^a. Therefore, two gauge transformations do not close in general !

To able to generalize to any gauge group *(J. Wess et.al. EPJ'01)*

- let the parameters to be in the enveloping algebra of the Lie algebra :

$$\hat{\Lambda} = \alpha_a T^a + \Lambda^1_{ab} : T^a T^b : + \cdots \Lambda^{n-1}_{a_1 \cdots a_n} : T^{a_1} \cdots T^{a_n} : + \cdots$$

- let all NC fields and parameters depend only on Lie algebra valued fields $A, \psi, \cdots$ and parameter α i.e.

$$\hat{A}_\mu \equiv \hat{A}_\mu(A) \quad , \quad \hat{\Psi}_\mu \equiv \hat{\Psi}_\mu(A, \psi) \quad , \quad \hat{\Lambda} = \hat{\Lambda}(A, \alpha)$$

- impose NC gauge consistency condition:

$$i\delta_\alpha \hat{\Lambda}_\beta - i\delta_\beta \hat{\Lambda}_\alpha - [\hat{\Lambda}_\alpha, \hat{\Lambda}_\beta]_* = i\hat{\Lambda}_{-i[\alpha,\beta]}.$$

Note that, above construction of Wess et.al. is obtained entirely independent of string theory !

In contrast to SW we now have an equation only for the gauge parameter

$$i\delta_\alpha \hat{\Lambda}_\beta - i\delta_\beta \hat{\Lambda}_\alpha - [\hat{\Lambda}_\alpha, \hat{\Lambda}_\beta]_* = i\hat{\Lambda}_{-i[\alpha,\beta]}.$$

and once we solve it we can then solve

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu(A;\theta) = \delta_\alpha \hat{A}_\mu(A;\theta)$$

only for A_μ.

To find the solutions we expand $\hat{\Lambda}_\alpha$ and $\hat{A}_\mu$ as formal power series,

$$\begin{aligned} \hat{\Lambda}_\alpha &= \alpha + \Lambda^1_\alpha + \cdots + \Lambda^n_\alpha + \cdots \,, \\ \hat{A}_\mu &= A_\mu + A^1_\mu + \cdots + A^n_\mu + \cdots \,, \end{aligned}$$

The superscript n denotes the order of θ.

NC gauge consistency condition and gauge equivalence condition

$$i\delta_\alpha\hat{\Lambda}_\beta - i\delta_\beta\hat{\Lambda}_\alpha - [\hat{\Lambda}_\alpha, \hat{\Lambda}_\beta]_* = i\hat{\Lambda}_{-i[\alpha,\beta]} \quad , \quad \hat{\delta}_{\hat{\Lambda}}\hat{A}_\mu(A;\theta) = \delta_\alpha\hat{A}_\mu(A;\theta)$$

can be written for the n–th order components of $\hat{\Lambda}_\alpha$ and $\hat{A}_\mu$ as

$$i\delta_\alpha\Lambda^n_\beta - i\delta_\beta\Lambda^n_\alpha - \sum_{p+q+r=n}[\Lambda^p_\alpha, \Lambda^q_\beta]_{*^r} = i\hat{\Lambda}^n_{-i[\alpha,\beta]}$$

$$\delta_\alpha A^n_\mu = \partial_\mu\Lambda^n_\alpha - i\sum_{p+q+r=n}[A^p_\mu\,,\,\Lambda^q_\alpha]_{*^r}$$

respectively. Here, $*^r$ denotes :

$$f(x) *^r g(x) \equiv \frac{1}{r!}\left(\frac{i}{2}\right)^r \theta^{\mu_1\nu_1}\cdots\theta^{\mu_r\nu_r}\partial_{\mu_1}\cdots\partial_{\mu_r}f(x)\partial_{\nu_1}\cdots\partial_{\nu_r}g(x).$$

We can rearrange the above Eq.s for any order n

$$\Delta\Lambda^n \equiv i\delta_\alpha\hat{\Lambda}^n_\beta - i\delta_\beta\hat{\Lambda}^n_\alpha - [\alpha, \hat{\Lambda}^n_\beta] - [\hat{\Lambda}^n_\alpha, \beta] - i\hat{\Lambda}^n_{-i[\alpha,\beta]} = \sum_{\substack{p+q+r=n,\\ p,q\neq n}} [\Lambda^p_\alpha, \Lambda^q_\beta]_{*^r}$$

$$\Delta_\alpha A^n_\mu \equiv \delta_\alpha A^n_\mu - i[\alpha, A^n_\mu] = \partial_\mu\Lambda^n_\alpha + i \sum_{\substack{p+q+r=n,\\ q\neq n}} [\Lambda^p_\alpha, A^q_\mu]_{*^r}$$

so that the l.h.s contains only the n-th order components.
However, note that one can extract the homogeneous parts such that

$$\Delta\tilde{\Lambda}^n_\alpha = 0 \quad , \quad \Delta_\alpha\tilde{A}^n_\mu = 0$$

It is clear that one can add any homogeneous solution $\tilde{\Lambda}^n_\alpha, \tilde{A}^n_\mu$ to the inhomogeneous solutions $\Lambda^n_\alpha, A^n_\mu$ with arbitrary coefficients.

First order solution given in the original paper (*SW, JHEP'99*) :

$$\Lambda^1_\alpha = -\frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda \alpha\}$$

$$A^1_\gamma = -\frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda A_\gamma + F_{\lambda\gamma}\}.$$

One can find the field strength form the definition :

$$F^1_{\gamma\rho} = -\frac{1}{4}\theta^{\kappa\lambda}\Big(\{A_\kappa, \partial_\lambda F_{\gamma\rho} + D_\lambda F_{\gamma\rho}\} - 2\{F_{\gamma\kappa}, F_{\rho\lambda}\}\Big).$$

- These solutions are not unique since one can add homogeneous solutions with arbitrary coefficients. i.e.

$$\tilde{\Lambda}^1 = ic_1\theta^{\mu\nu}[A_\mu, \partial_\nu \alpha] \quad , \quad \tilde{A}^1_\rho = c_2\theta^{\mu\nu} D_\rho F_{\mu\nu}$$

Therefore,

$$A^1_\rho + ic_1\theta^{\mu\nu}([\partial_\rho A_\mu, A_\nu] - i[[A_\rho, A_\mu], A_\nu]) + c_2\theta^{\mu\nu} D_\rho F_{\mu\nu}$$

is also a solution

2nd order solutions (*Moller'04*) A^2_μ and Λ^2_α can be written in terms of lower order solutions : (*K.Ü ,B. Yapiskan PRD'08.*)

$$\Lambda^2_\alpha = -\frac{1}{8}\theta^{\kappa\lambda}\left(\{A^1_\kappa, \partial_\lambda \alpha\} + \{A_\kappa, \partial_\lambda \Lambda^1_\alpha\}\right) - \frac{i}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu A_\kappa, \partial_\nu \partial_\lambda \alpha]$$

$$\begin{aligned} A^2_\gamma &= -\frac{1}{8}\theta^{\kappa\lambda}\left(\{A^1_\kappa, \partial_\lambda A_\gamma + F_{\lambda\gamma}\} + \{A_\kappa, \partial_\lambda A^1_\gamma + F^1_{\lambda\gamma}\}\right) \\ &\quad -\frac{i}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu A_\kappa, \partial_\nu(\partial_\lambda A_\gamma + F_{\lambda\gamma})]. \end{aligned}$$

The field strength at the second order can also be written in terms of first order solutions :

$$\begin{aligned} F^2_{\gamma\rho} &= -\frac{1}{8}\theta^{\kappa\lambda}\Big(\{A_\kappa, \partial_\lambda F^1_{\gamma\rho} + (D_\lambda F_{\gamma\rho})^1\} \\ &\quad + \{A^1_\kappa, \partial_\lambda F_{\gamma\rho} + D_\lambda F_{\gamma\rho}\} - 2\{F_{\gamma\kappa}, F^1_{\rho\lambda}\} - 2\{F^1_{\gamma\kappa}, F_{\rho\lambda}\}\Big) \\ &\quad -\frac{i}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}\Big([\partial_\mu A_\kappa, \partial_\nu(\partial_\lambda F_{\gamma\rho} + D_\lambda F_{\gamma\rho})] - 2[\partial_\mu F_{\gamma\kappa}, \partial_\nu F_{\rho\lambda}]\Big). \end{aligned}$$

Where, $(D_\lambda F_{\gamma\rho})^1 = D_\lambda F^1_{\gamma\rho} - i[A^1_\lambda, F_{\gamma\rho}] + \frac{1}{2}\theta^{\mu\nu}\{\partial_\mu A_\lambda, \partial_\nu F_{\gamma\rho}\}$.

By analyzing first two order solutions one can conjecture the general structure (*K.Ü ,B. Yapiskan PRD'08.*) :

$$\Lambda_\alpha^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} \{A_\kappa^p, \partial_\lambda \Lambda_\alpha^q\}_{*^r}$$

$$A_\gamma^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} \{A_\kappa^p, \partial_\lambda A_\gamma^q + F_{\lambda\gamma}^q\}_{*^r}.$$

The overall constant $-1/4(n+1)$ is **fixed uniquely** with third order solutions.

Solution of Seiberg-Witten Differential Equation

Let us vary the deformation parameter infinitesimally

$$\theta \to \theta + \delta\theta$$

To get equivalent physics, $\hat{A}(\theta)$ and $\hat{\Lambda}(\theta)$ should change when θ is varied, *(SW'99)*:

$$\begin{aligned} \delta\hat{A}_\gamma(\theta) &= \hat{A}_\gamma(\theta+\delta\theta) - \hat{A}_\gamma(\theta) \\ &= \delta\theta^{\mu\nu}\frac{\partial\hat{A}_\gamma}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\kappa\lambda}\{\hat{A}_\kappa, \partial_\lambda\hat{A}_\gamma + \hat{F}_{\lambda\gamma}\}_* \end{aligned}$$

$$\begin{aligned} \delta\hat{\Lambda}(\theta) &= \hat{\Lambda}(\theta+\delta\theta) - \hat{\Lambda}(\theta) \\ &= \delta\theta^{\mu\nu}\frac{\partial\hat{\Lambda}}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\theta^{\kappa\lambda}\{\hat{A}_\kappa, \partial_\lambda\hat{\Lambda}\}_* \end{aligned} \quad (1)$$

These differential equations are commonly called SW differential equations.

To find solutions of the differential equation

- expand NC gauge parameter and NC gauge field into a Taylor series:

$$\begin{aligned}\hat{\Lambda}_{\alpha}^{(n)} &= \alpha + \Lambda_{\alpha}^{1} + \cdots + \Lambda_{\alpha}^{n},\\ \hat{A}_{\mu}^{(n)} &= A_{\mu} + A_{\mu}^{1} + \cdots + A_{\mu}^{n}.\end{aligned}$$

  here $\hat{\Lambda}_{\alpha}^{(n)}$ and $\hat{A}_{\mu}^{(n)}$ denotes the sum up to order n !
- Then it is possible to write *(Wulkenhaar et.al'01)* :

$$\hat{\Lambda}_{\alpha}^{(n+1)} = \alpha - \frac{1}{4}\sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\cdots\theta^{\mu_k\nu_k}\left(\frac{\partial^{k-1}}{\partial\theta^{\mu_2\nu_2}\cdots\partial\theta^{\mu_k\nu_k}}\{\hat{A}_{\mu_1}^{(k)}, \partial_{\nu_1}\hat{\Lambda}_{\alpha}^{(k)}\}_*\right)_{\theta=0}$$

$$\hat{A}_{\gamma}^{(n+1)} = A_{\gamma} - \frac{1}{4}\sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\cdots\theta^{\mu_k\nu_k}\left(\frac{\partial^{k-1}}{\partial\theta^{\mu_2\nu_2}\cdots\partial\theta^{\mu_k\nu_k}}\{\hat{A}_{\mu_1}^{(k)}, \partial_{\nu_1}\hat{A}_{\gamma}^{(k)} + \hat{F}_{\nu_1\gamma}^{(k)}\}_*\right)_{\theta}$$

Contrary to the solutions presented in the previous section, these expressions explicitly contain

- derivatives w.r.t. θ
- $*$–product itself
- and given as a sum of all (n+1) orders.

However, it is possible to extract our recursive solutions from the above expressions.

Let us write the $n+1$–st component of $\hat{\Lambda}_\alpha^{(n+1)}$:

$$\Lambda_\alpha^{n+1} = -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\dots\partial\theta^{\mu_n\nu_n}}\{\hat{A}_{\mu_1}^{(n)},\partial_{\nu_1}\hat{\Lambda}_\alpha^{(n)}\}_*\right)_{\theta=0}$$

Since, θ is set to zero after taking the derivatives, the expression in the paranthesis can be written as a sum up to n-th order:

$$\Lambda_\alpha^{n+1} = -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\dots\partial\theta^{\mu_n\nu_n}}\sum_{p+q+r=n}\{A_\mu^p,\partial_\nu\Lambda_\alpha^q\}_{*^r}\right).$$

It is then an easy exercise to show that the above equation reduces to the recursive formula :

$$\Lambda_\alpha^{n+1} = -\frac{1}{4(n+1)}\theta^{\mu\nu}\sum_{p+q+r=n}\{A_\mu^p,\partial_\nu\Lambda_\alpha^q\}_{*^r}.$$

With the same algebraic manipulation one can also derive the same recursive formula for the gauge field

$$
\begin{aligned}
A_\gamma^{n+1} &= -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\times \\
&\qquad \times\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\ldots\partial\theta^{\mu_n\nu_n}}\{\hat{A}_{\mu_1}^{(n)},\partial_{\nu_1}\hat{A}_\gamma^{(n)}+\hat{F}_{\nu_1\gamma}^{(n)}\}_*\right)_{\theta=0} \\
&= -\frac{1}{4(n+1)}\theta^{\mu\nu}\sum_{p+q+r=n}\{A_\mu^p,\partial_\nu A_\gamma^q+F_{\nu\gamma}^q\}_{*^r}.
\end{aligned}
$$

Seiberg–Witten Map for Matter Fields

SW–map of a NC field $\hat{\Psi}$ in a gauge invariant theory can be derived from *(J. Wess et.al. EPJ'01)*:

$$\hat{\delta}_{\hat{\Lambda}} \hat{\Psi}(\psi, A; \theta) = \delta_{\alpha} \hat{\Psi}(\psi, A; \theta).$$

The solution of this gauge equivalence relation can be found order by order by after expanding the NC field $\hat{\Psi}$ as formal power series in θ

$$\hat{\Psi} = \psi + \hat{\Psi}^1 + \cdots + \hat{\Psi}^n + \cdots$$

Fundamental Representation :

Ordinary gauge transformation of ψ is written as

$$\delta_\alpha \psi = i\alpha\psi.$$

NC gauge transformation is defined with the help of $*$–product :

$$\hat{\delta}_{\hat{\Lambda}} \hat{\Psi} = i\hat{\Lambda}_\alpha * \hat{\Psi}.$$

Following the general strategy the gauge equivalence relation reads

$$\Delta_\alpha \Psi^n \equiv \delta_\alpha \Psi^n - i\alpha\Psi^n = i \sum_{\substack{p+q+r=n, \\ q \neq n}} \Lambda_\alpha^p *^r \Psi^q,$$

for all orders.

As discussed before, one is free to add any homogeneous solution $\tilde{\Psi}^n$ of the equation

$$\Delta_\alpha \tilde{\Psi}^n = 0$$

to the solutions Ψ^n.

A solution for the first order is given by Wess et.al :

$$\Psi^1 = -\frac{1}{4}\theta^{\kappa\lambda}A_\kappa(\partial_\lambda + D_\lambda)\psi$$

where $D_\mu\psi = \partial_\mu\psi - iA_\mu\psi$.

The SW differential equation from the first order solution reads :

$$\delta\theta^{\mu\nu}\frac{\partial\hat{\Psi}}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\kappa\lambda}\hat{A}_\kappa * (\partial_\lambda\hat{\Psi} + \hat{D}_\lambda\hat{\Psi})$$

which can also be written as

$$\frac{\partial\hat{\Psi}}{\partial\theta^{\kappa\lambda}} = -\frac{1}{8}\hat{A}_\kappa * (\partial_\lambda\hat{\Psi} + \hat{D}_\lambda\hat{\Psi}) + \frac{1}{8}\hat{A}_\lambda * (\partial_\kappa\hat{\Psi} + \hat{D}_\kappa\hat{\Psi})$$

The NC covariant derivative here is defined as :

$$\hat{D}_\mu\hat{\Psi} = \partial_\mu\hat{\Psi} - i\hat{A}_\mu * \hat{\Psi} .$$

By using a similar method, we expand $\hat{\Psi}$ in Taylor series :

$$
\begin{aligned}
\hat{\Psi}^{(n+1)} &= \psi + \Psi^1 + \Psi^2 + \cdots + \Psi^{n+1} \\
&= \psi + \sum_{k=1}^{n+1} \frac{1}{k!} \theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\ldots\theta^{\mu_k\nu_k} \left(\frac{\partial^k}{\partial\theta^{\mu_1\nu_1}\ldots\partial\theta^{\mu_k\nu_k}} \left(\hat{\Psi}^{(n+1)} \right) \right)_{\theta=0}
\end{aligned}
$$

From the differential equation we get

$$
\begin{aligned}
\hat{\Psi}^{(n+1)} &= \psi - \frac{1}{4} \sum_{k=1}^{n+1} \frac{1}{k!} \theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\ldots\theta^{\mu_k\nu_k} \times \\
&\qquad \times \Big(\frac{\partial^{k-1}}{\partial\theta^{\mu_2\nu_2}\ldots\partial\theta^{\mu_k\nu_k}} \hat{A}^{(k)}_{\mu_1} * (\partial_{\nu_1}\hat{\Psi}^{(k)} + (\hat{D}_{\nu_1}\hat{\Psi})^{(k)}) \Big)_{\theta=0}
\end{aligned}
$$

where

$$
(\hat{D}_\mu \hat{\Psi})^{(n)} = \partial_\mu \hat{\Psi}^{(n)} - i\hat{A}^{(n)}_\mu * \hat{\Psi}^{(n)}.
$$

For the abelian case this solution differs from the one given in Wulkenhaar et.al. by a homogeneous solution.

To find the all order recursive solution we write the $n+1$–st component :

$$\Psi^{n+1} = -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n} \times \times \left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\ldots\partial\theta^{\mu_n\nu_n}}\hat{A}_\mu^{(n)} * (\partial_\nu\hat{\Psi}^{(n)} + (\hat{D}_\nu\hat{\Psi})^{(n)})\right)_{\theta=0}$$

$$= -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n} \times \times \left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\ldots\partial\theta^{\mu_n\nu_n}}\sum_{p+q+r=n} A_\mu^p *^r (\partial_\nu\Psi^{(q)} + (D_\nu\Psi)^q)\right)$$

where

$$(D_\mu\Psi)^n = \partial\Psi^n - i\sum_{p+q+r=n} A_\mu^p *^r \Psi^q.$$

After taking derivatives w.r.t. θ's we obtain the all order recursive solution of the gauge equivalence relation :

$$\psi^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} A_{\kappa}^{p} *^{r} (\partial_{\lambda}\Psi^{(q)} + (D_{\lambda}\Psi)^{q}).$$

- The first order solution of Wess et.al. is obtained by setting $n = 0$.
- By setting $n = 1$ we find the second order solution of Möller .

Adjoint representation :

The gauge transformation reads as

$$\delta_\alpha \psi = i[\alpha, \psi]$$

Non–commutative generalization can be written as

$$\hat{\delta}_{\hat{\Lambda}} \hat{\Psi} = i[\hat{\Lambda}_\alpha, \hat{\Psi}]_* \, .$$

Following the general strategy, the gauge equivalence relation can be written as

$$\Delta_\alpha \Psi^n := \delta_\alpha \Psi^n - i[\alpha, \Psi] = i \sum_{\substack{p+q+r=n, \\ q \neq n}} [\Lambda_\alpha^p, \Psi^q]_{*^r}$$

for all orders. The solutions can be found

- either by directly solving this equation order by order
- by solving the respective differential equation
- However, possibly the easiest way to obtain the solution is to use dimensional reduction. (K.U, Saka PRD'07)

By setting the components of the deformation parameter θ on the compactified dimensions zero, i.e.

$$\Theta^{MN} = \begin{pmatrix} \theta^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix},$$

the trivial dimensional reduction (i.e. from six to four dimensions) leads to the general n–th order solution for a complex scalar field :

$$\psi^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} \{A^p_\kappa, (\partial_\lambda \Psi^{(q)} + (D_\lambda \Psi)^q)\}_{*^r}.$$

Since the structure of the solutions are the same for both the scalar and fermionic fields, this solution can also be used for the fermionic fields.

Note that, after introducing the anticommutators / commutators properly, the form of the solution is similar with the one given for the fundamental case except that now

$$D_\mu\psi = \partial_\mu\psi - i[A_\mu, \psi]$$

and hence

$$(D_\mu\Psi)^n = \partial_\mu\Psi^n - i\sum_{p+q+r=n}[A^p_\mu, \Psi^q]_{*^r}\ .$$

The same result can also be obtained by solving the differential equation :

$$\frac{\partial\hat{\Psi}}{\partial\theta^{\kappa\lambda}} = -\frac{1}{8}\{\hat{A}_\kappa, (\partial_\lambda\hat{\Psi} + \hat{D}_\lambda\hat{\Psi})\}_* + \frac{1}{8}\{\hat{A}_\lambda, (\partial_\kappa\hat{\Psi} + \hat{D}_\kappa\hat{\Psi})\}_*.$$

Therefore, the result obtained above via dimensional reduction can also be thought as an independent check of the aforementioned results.

CONCLUSION

Equations,

$$\hat{A}_\mu(A;\theta) + \hat{\delta}_{\hat{\Lambda}}\hat{A}_\mu(A;\theta) = \hat{A}_\mu(A+\delta_\alpha A;\theta).$$

$$\hat{\Psi}(A,\psi;\theta) + \hat{\delta}_{\hat{\Lambda}}\hat{\Psi}(A,\psi;\theta) = \hat{\Psi}(A+\delta_\alpha A,\psi+\delta_\alpha\psi;\theta).$$

can be solved as

$$\Lambda_\alpha^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}\{A_\kappa^p, \partial_\lambda\Lambda_\alpha^q\}_{*^r}$$

$$A_\gamma^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}\{A_\kappa^p, \partial_\lambda A_\gamma^q + F_{\lambda\gamma}^q\}_{*^r}.$$

$$\psi^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}A_\kappa^p *^r(\partial_\lambda\Psi^{(q)} + (D_\lambda\Psi)^q).$$