On the All Order Solutions of Seiberg-Witten Map for Noncommutative Gauge Theories[1]

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[1] **K. Ulker, B Yapiskan** Phys. Rev. D **77**, 065006 (2008) [arXiv:0712.0506]
K. Ulker [arXiv:1201.2192]

Non–Commutativity

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- Consider quantum mechanics, we have Heisenberg uncertainty relation
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i.e. the concept of point in phase space looses its meaning !
- Note that in QM x and p are self-adjoint *"non-commutative"* operators satisfying :
$$[\hat{x}, \hat{p}] = i\hbar$$

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  (See for instance Doplicher et.al CMP'95)

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- Now, assume that space-time itself is non-commutative at some very small scale.
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  (See for instance Doplicher et.al CMP'95)
- The simplest NC space that is extensively studied in the literature is the deformation of D–dimensional Minkowski or Euclidean space $\mathbb{R}^D$:

$$[\hat{x}^\mu \, , \, \hat{x}^\nu] = i\theta^{\mu\nu}$$

with the help of a real constant antisymmetric parameter Θ.

Possible applications :

- Solid state physics (QHE, Graphen, Topological Insulators etc.)
- Extension of Standard Model.
- Neutrinos faster than light (?!?!)
- But maybe the most interesting is the possibility to construct the quantum theory of gravity since the noncommutativity in space–time induces naturally a quantum structure.

reviews: Douglas–Nekrasov [hep-th/0106048], Szabo [hep-th/0109162], Martin [1101.4783], Rivelles [1101.4579]

Maybe the simplest physical motivation is a charged particle moving in the plane (x^1, x^2) and in the presence of a constant, perpendicular magnetic field B :

$$L = \frac{m}{2}\dot{x}^i\dot{x}^i - \frac{B}{2}\epsilon_{ij}\dot{x}^i x^j$$

Assume that B is strong such that $|m\dot{x}^i| << |B_{ij}x^j|$ we get,

$$p_i = -B\epsilon_{ij}x^j$$

which leads to space non-commutativity :

$$[\hat{x}^i, \hat{x}^j] = \frac{i}{B}\epsilon^{ij} \equiv \theta^{ij}$$

- A similar relation can be obtained from string theory (Seiberg-Witten JHEP'99)
- But the relation to string theory leads to the interesting result that *certain NC gauge theories can be mapped to commutative ones.*

$$\hat{A}^{\mu} \longrightarrow \hat{A}_{\mu}(A_{\mu}, \theta)$$

  which leads for instance to

$$S_{NC-YM}[\hat{A}] \longrightarrow S_{YM}[A] + S_{\theta}[A, \theta]$$

- This map is commonly called Seiberg–Witten (SW) map.
- The aim of this talk is to present how to construct the explicit solutions of this map to all orders in the non-commutativity parameter θ.

Outline

- Moyal $*$–product.
- Non-commutative Yang–Mills $(\hat{A}\,,\,\hat{\Lambda})$
- Seiberg – Witten Map $(\hat{A} \to \hat{A}(A,\theta)\,,\,\hat{\Lambda} \to \hat{\Lambda}(A,\alpha,\theta))$
- Construction of the maps order by order leads to all order solutions
- All order solutions can be obtained from SW differential equation
- SW map of other fields $(\hat{\Psi} \to \hat{\Psi}(A,\psi,\theta))$
- Conclusion

Groenewold–Moyal $*$-product

- The simplest way to introduce non-commutativity is to use (Groenewold–Moyal) $*$–product

$$f(x) * g(x) \equiv exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)|_{y\to x}$$
$$= f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \cdots$$

for *real CONSTANT antisymmetric parameter* θ !

- The commutation relation can now be written as a $*$–commutator of the ordinary coordinates :

$$[x^\mu\,,\,x^\nu]_* \equiv x^\mu\,*\,x^\nu - x^\nu\,*\,x^\mu = i\theta^{\mu\nu}.$$

- $*$–product of two functions is obviously non–commutative :

$$f(x) * g(x) \neq g(x) * f(x)$$

However, under the integral sign we can write

$$\int f * g = \int g * f = \int f . g$$

for functions f and g that vanish rapidly enough at infinity.

- $*$–product is associative :

$$f(x) * g(x) * h(x) = \big(f * g\big) * h = f * \big(g * h\big)$$

And similarly under the integral sign we can get rid of one $*$

$$\int f * g * h = \big(f * g\big) . h(x) = f . \big(g * h\big)$$

NC Yang–Mills Theory

The simplest NC QFT models can then be obtained by replacing the ordinary product with the $*$–product !

- For instance the action of NC YM theory is written as :

$$\hat{S} = -\frac{1}{4} Tr \int d^4x \hat{F}^{\mu\nu} * \hat{F}_{\mu\nu} = -\frac{1}{4} Tr \int d^4x \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}$$

  where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_*$$

  is the NC field strength of the NC gauge field $\hat{A}$.

- The action is invariant under the NC gauge transformations:

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu = \partial_\mu \hat{\Lambda} - i[\hat{A}_\mu, \hat{\Lambda}]_* \equiv \hat{D}_\mu \hat{\Lambda} \quad \Rightarrow \quad \hat{\delta}_{\hat{\Lambda}} \hat{F}_{\mu\nu} = i[\hat{\Lambda}, \hat{F}_{\mu\nu}]_* .$$

  Here, $\hat{\Lambda}$ is the NC gauge parameter.

Seiberg–Witten Map

One can derive both conventional and NC gauge theories from string theory by using different regularization procedures *(SW'99)*.

Let A_μ and α be the ordinary counterparts of $\hat{A}_\mu$ and $\hat{\Lambda}$ respectively.

$\Longrightarrow$ There must be a map from a commutative gauge field A to a noncommutative one $\hat{A}$, that arises from the requirement that *gauge invariance should be preserved* !

$$\hat{A}(A) + \hat{\delta}_{\hat{\Lambda}} \hat{A}(A) = \hat{A}(A + \delta_\alpha A)$$

where δ_α is the ordinary gauge transformation :

$$\delta_\alpha A_\mu = \partial_\mu \alpha - i[A_\mu, \alpha] \equiv D_\mu \alpha.$$

- At first sight it seems that a field redefinition i.e. $\hat{A} = \hat{A}(A, \partial A, \cdots ; \theta)$ and $\hat{\Lambda} = \hat{\Lambda}(\Lambda, \partial\Lambda, \cdots ; \theta)$ would do the job.
- But this is not true ! Consider $U(1)$. We have $\delta A_\mu = \partial_\mu \alpha$ for the commutative case and $\delta \hat{A}_\mu = \partial_\mu \hat{\Lambda} - i[\hat{A}, \Lambda]_*$ for the non-commutative case. One is abelian and the other is non-abelian. An abelian group cannot be isomorphic to a nonabelian group !

However, to do physics, we only need to know when two gauge fields A and A' should be considered gauge-equivalent.

- If $A = UA'U^{-1}$ where $U = e^{(i\alpha)}$ then for the corresponding NC fields we must have $\hat{A} = \hat{U}\hat{A}'\hat{U}^{-1}$ where $U = e^{(i\hat{\Lambda})}$
- Therefore, if $\hat{\Lambda}$ depends also on the gauge field i.e.

$$\hat{\Lambda} = \hat{\Lambda}(\alpha, A, \theta)$$

we do not get any well-defined mapping between the gauge groups,
- but we get an identification only of the gauge equivalence relations in the following sense :

$$\hat{A}(A) + \hat{\delta}_{\hat{\Lambda}}\hat{A}(A) = \hat{A}(A + \delta_\alpha A)$$

where δ_α is the ordinary gauge transformation :

$$\delta_\alpha A_\mu = \partial_\mu \alpha - i[A_\mu, \alpha] \equiv D_\mu \alpha.$$

This map can be rewritten as

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu(A;\theta) = \hat{A}_\mu(A + \delta_\alpha A;\theta) - \hat{A}_\mu(A;\theta) = \delta_\alpha \hat{A}_\mu(A;\theta)$$

Since SW map imposes the following functional dependence :

$$\hat{A}_\mu = \hat{A}_\mu(A;\theta) \quad , \quad \hat{\Lambda} = \hat{\Lambda}_\alpha(\alpha, A;\theta).$$

$\Longrightarrow$ one has to solve

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu(A;\theta) = \delta_\alpha \hat{A}_\mu(A;\theta)$$

simultaneously for $\hat{A}_\mu$ and $\hat{\Lambda}_\alpha$ and it is difficult !

Now, remember the ordinary gauge consistency condition

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{-i[\alpha,\beta]}.$$

(check for instance for $\delta_\alpha \psi = i\alpha\psi$)
Let, Λ be a Lie algebra valued gauge parameter $\Lambda = \Lambda_a T^a$. For the non-commutative case, we get

$$(\delta_{\Lambda_\alpha}\delta_{\Lambda_\beta} - \delta_{\Lambda_\beta}\delta_{\Lambda_\alpha})\hat{\Psi} = \frac{1}{2}[T^a, T^b]\{\Lambda_{\alpha,a}, \Lambda_{\beta,b}\}_* * \hat{\Psi} + \frac{1}{2}\{T^a, T^b\}[\Lambda_{\alpha,a}, \Lambda_{\beta,b}]_* * \hat{\Psi}$$

Only a $U(N)$ gauge theory allows to express $\{T^a, T^b\}$ again in terms of T^a. Therefore, two gauge transformations do not close in general !

To be able to generalize to any gauge group *(J. Wess et.al. EPJ'01)*

- let the parameters be in the enveloping algebra of the Lie algebra :

$$\hat{\Lambda} = \lambda_a T^a + \Lambda^1_{ab} : T^a T^b : + \cdots \Lambda^{n-1}_{a_1 \cdots a_n} : T^{a_1} \cdots T^{a_n} : + \cdots$$

- let all NC fields and parameters depend only on Lie algebra valued fields $A, \psi, \cdots$ and parameter α i.e.

$$\hat{A}_\mu \equiv \hat{A}_\mu(A) \quad , \quad \hat{\Psi}_\mu \equiv \hat{\Psi}_\mu(A, \psi) \quad , \quad \hat{\Lambda} = \hat{\Lambda}(A, \alpha)$$

- impose NC gauge consistency condition:

$$i\delta_\alpha \hat{\Lambda}_\beta - i\delta_\beta \hat{\Lambda}_\alpha - [\hat{\Lambda}_\alpha, \hat{\Lambda}_\beta]_* = i\hat{\Lambda}_{-i[\alpha,\beta]}.$$

Note that, above construction of Wess et.al. is obtained entirely independent of string theory !

In contrast to SW we now have an equation only for the gauge parameter

$$i\delta_\alpha \hat{\Lambda}_\beta - i\delta_\beta \hat{\Lambda}_\alpha - [\hat{\Lambda}_\alpha, \hat{\Lambda}_\beta]_* = i\hat{\Lambda}_{-i[\alpha,\beta]}.$$

and once we solve it we can then solve

$$\hat{\delta}_{\hat{\Lambda}} \hat{A}_\mu(A;\theta) = \delta_\alpha \hat{A}_\mu(A;\theta)$$

only for A_μ.

BRST transformations

- gauge parameter $\alpha \to c$ ghost field.
- gauge transformation $\delta \to s$ BRST transformation.

$$sA_\mu = D_\mu c \quad , \quad sc = ic \cdot c$$

Note that c is a Grassmann valued (anti–commuting) Fadeev–Popov ghost.

Therefore, the operator s is an odd superderivation of ghost number one

$$s(f.g) = (sf).g \pm f.(sg) \quad , \quad s^2 = 0$$

that commutes with ordinary derivatives

$$s\partial_\mu = \partial_\mu s$$

NC–BRST transformations

Generalize $s \longrightarrow \hat{s}$ (Zumino etal'01):

$$\hat{s}\hat{A}_\mu = \hat{D}_\mu \hat{C} \quad , \quad \hat{s}\hat{\Psi} = i\hat{C} * \hat{\Psi} \quad , \quad \hat{s}\hat{C} = i\hat{C} * \hat{C}$$

where $\hat{C}$ is the NC counterpart of c.

As s, the NC BRST transformation $\hat{s}$ is nilpotent

$$\hat{s}^2 = 0 \tag{1}$$

and therefore one can explicitly study the underlying cohomological structure in the NC case.

Moreover, by requiring that the SW–map respects the gauge equivalence

$$\hat{s}\hat{A}_\mu(A;\theta) = s\hat{A}_\mu(A;\theta). \quad , \quad \hat{s}\hat{\Psi}(\psi, A;\theta) = s\hat{\Psi}(\psi, A;\theta) \tag{2}$$

it can be shown that the nilpotency of $\hat{s}$ is nothing but the gauge consistency condition.

To find the solutions of SW–map expand $\hat{C}$ and $\hat{A}_\mu$ as formal power series,

$$\begin{aligned}\hat{A}_\mu &= A_\mu + A_\mu^{(1)} + \cdots + A_\mu^{(n)} + \cdots \\ \hat{C} &= c + C^{(1)} + \cdots + C^{(n)} + \cdots\end{aligned}$$

The superscript n denotes the order of θ.

At each order in θ we get

$$\begin{aligned}sC^{(n)} &= i \sum_{p+q+r=n} C^{(p)} *^r C^{(q)} \\ sA_\mu^{(n)} &= \partial_\mu C_\alpha^{(n)} - i \sum_{p+q+r=n} [A_\mu^{(p)}, C_\alpha^{(q)}]_{*^r}\end{aligned}$$

where

$$f(x) *^r g(x) \equiv \frac{1}{r!}\left(\frac{i}{2}\right)^r \theta^{\mu_1\nu_1} \cdots \theta^{\mu_r\nu_r} \partial_{\mu_1} \cdots \partial_{\mu_r} f(x) \partial_{\nu_1} \cdots \partial_{\nu_r} g(x)$$

We can rearrange the above Eq.s for any order n

$$\Delta C^{(n)} \equiv sC^{(n)} - i\{c, C^{(n)}\} \quad = \quad i \sum_{\substack{p+q+r=n,\\ p,q\neq n}} C^{(p)} *^r C^{(q)}$$

$$\Delta A_\mu^{(n)} \equiv sA_\mu^{(n)} - i[c, A_\mu^{(n)}] \quad = \quad \partial_\mu C_\alpha^{(n)} - i \sum_{\substack{p+q+r=n,\\ p\neq n}} [A_\mu^{(p)}, C_\alpha^{(q)}]_{*^r}$$

so that the l.h.s contains only the n-th order components.

Δ is also nilpotent

$$\Delta^2 = 0$$

At each order θ we have a set of inhomogeneous equations

$$\Delta C^{(n)} = \mathcal{G}^{(n)}(\theta^n; c, A) \Rightarrow \Delta \mathcal{G}^{(n)} = 0$$

$$\Delta A_\mu^{(n)} = \mathcal{H}^{(n)}(\theta^n; A) \Rightarrow \Delta \mathcal{H}^{(n)} = 0$$

that gives the solution of SW–map.

However, one can extract the homogeneous parts such that

$$\Delta \tilde{C}^{(n)} = 0 \quad , \quad \Delta \tilde{A}_\mu^{(n)} = 0$$

It is clear that one can add any homogeneous solutions $\tilde{C}^{(n)}, \tilde{A}_\mu^{(n)}$ to the inhomogeneous solutions $C^{(n)}, A_\mu^{(n)}$ with arbitrary coefficients. This is the ambiguity in the SW–map.

To find 1st and 2nd order solutions of the SW map, the following strategy was used :

- Determine the dimension and the index structure of the solution first.
- Write the most general expression in terms of fields and their derivatives satisfying these constraints.
- Fix the coefficients by plugging these expressions in the respective equivalence or consistency conditions

However, note that

- This strategy is difficult for higher orders !!!
- All these maps are different from each other up to a homogeneous solution with different coefficients.
- 2nd order solutions that were found in the literature are too long expressions and so difficult to use in practical calculations.

On the other hand, gauge consistency condition and gauge equivalence conditions are given as recursive relations between the lower order solutions and higher order ones.

- t is natural to ask whether the solutions of these equations can also be written in a recursive way or not.
- If this is possible, is it possible to find even higher order solutions ?

Why we need higher order solutions

- To study the consistency of the NC theory itself, such as renormalizability
- For NC gravity the first order contributions in θ from the SW–map vanish identically. Therefore, one has to know at least the second order solutions that are practically usable.

First order solution given in the original paper (*SW, JHEP'99*) :

$$C^{(1)} = -\frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda \alpha\}$$

$$A^{(1)}_\gamma = -\frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda A_\gamma + F_{\lambda\gamma}\}.$$

From the definition $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_*$ we get

$$F^{(1)}_{\gamma\rho} = -\frac{1}{4}\theta^{\kappa\lambda}\Big(\{A_\kappa, \partial_\lambda F_{\gamma\rho} + D_\lambda F_{\gamma\rho}\} - 2\{F_{\gamma\kappa}, F_{\rho\lambda}\}\Big).$$

These solutions are not unique since one can add homogeneous solutions with arbitrary coefficients. i.e.

$$\tilde{C}^{(1)} = i l_1 \theta^{\mu\nu}[A_\mu, \partial_\nu c] \quad , \quad \tilde{A}^{(1)}_\rho = l_2 \theta^{\mu\nu} D_\rho F_{\mu\nu}$$

2nd order solutions (*Moller'04*) $C^{(2)}$ and $A_\mu^{(2)}$

$$
\begin{aligned}
C^{(2)} &= \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}(\{A_\mu,\{\partial_\nu A_\kappa,\partial_\lambda c\}\} + \{A_\mu,\{A_\kappa,\partial_\nu\partial_\lambda c\}\} + \{\{A_\mu,\partial_\nu A_\kappa\},\partial_\lambda c\} \\
&- \{\{F_{\mu\kappa},A_\nu\},\partial_\lambda c\} - 2i[\partial_\mu A_\kappa,\partial_\nu\partial_\lambda c]).
\end{aligned}
$$

$$
\begin{aligned}
A_\gamma^{(2)} &= \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\{\{A_\kappa,\partial_\lambda A_\mu\},\partial_\nu A_\gamma\} - \{\{F_{\kappa\mu},A_\lambda\},\partial_\nu A_\gamma\} - 2i[\partial_\kappa A_\mu,\partial_\lambda\partial_\nu A_\gamma] \\
&-\{A_\mu,\{\partial_\nu F_{\kappa\gamma},A_\lambda\}\} - \{A_\mu,\{F_{\kappa\gamma},\partial_\nu A_\lambda^0\}\} + \{A_\mu,\{\partial_\nu A_\kappa,\partial_\lambda A_\gamma\}\} \\
&+\{A_\mu,\{A_\kappa,\partial_\nu\partial_\lambda A_\gamma\}\} - \{\{A_\kappa,\partial_\lambda F_{\mu\gamma}\},A_\nu\} + \{\{D_\kappa F_{\mu\gamma},A_\lambda\},A_\nu\} \\
&+2\{\{F_{\mu\kappa},F_{\gamma\lambda}\},A_\nu\} + 2i[\partial_\kappa F_{\mu\gamma},\partial_\lambda A_\nu] - \{F_{\mu\gamma},\{A_\kappa,\partial_\lambda A_\nu\}\} + \{F_{\mu\gamma},\{F_{\kappa\nu},A_\lambda\}\}\Big).
\end{aligned}
$$

can be written in terms of lower order solutions : (*K.Ü ,B. Yapiskan PRD'08.*)

$$
C^{(2)} = -\frac{1}{8}\theta^{\kappa\lambda}\left(\{A_\kappa^{(1)},\partial_\lambda c\} + \{A_\kappa,\partial_\lambda C^{(1)}\}\right) - \frac{i}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu A_\kappa,\partial_\nu\partial_\lambda c]
$$

$$
\begin{aligned}
A_\gamma^{(2)} &= -\frac{1}{8}\theta^{\kappa\lambda}\left(\{A_\kappa^{(1)},\partial_\lambda A_\gamma + F_{\lambda\gamma}\} + \{A_\kappa,\partial_\lambda A_\gamma^{(1)} + F_{\lambda\gamma}^{(1)}\}\right) \\
&\quad -\frac{i}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}[\partial_\mu A_\kappa,\partial_\nu(\partial_\lambda A_\gamma + F_{\lambda\gamma})].
\end{aligned}
$$

By analyzing first two order solutions one can conjecture the general structure (*K.Ü ,B. Yapiskan PRD'08.*) :

$$\Lambda_\alpha^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} \{A_\kappa^p, \partial_\lambda \Lambda_\alpha^q\}_{*^r}$$

$$A_\gamma^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} \{A_\kappa^p, \partial_\lambda A_\gamma^q + F_{\lambda\gamma}^q\}_{*^r}.$$

The overall constant $-1/4(n+1)$ is **fixed uniquely** with third order solutions.

These solutions can also be obtained from the SW–differential equation.

Solution of Seiberg-Witten Differential Equation

Let us vary the deformation parameter infinitesimally

$$\theta \to \theta + \delta\theta$$

To get equivalent physics, $\hat{A}(\theta)$ and $\hat{C}(\theta)$ should change when θ is varied, *(SW'99)*:

$$\begin{aligned}
\delta \hat{A}_\gamma(\theta) &= \hat{A}_\gamma(\theta + \delta\theta) - \hat{A}_\gamma(\theta) \\
&= \delta\theta^{\mu\nu} \frac{\partial \hat{A}_\gamma}{\partial \theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\kappa\lambda}\{\hat{A}_\kappa, \partial_\lambda \hat{A}_\gamma + \hat{F}_{\lambda\gamma}\}_*
\end{aligned}$$

$$\begin{aligned}
\delta \hat{C}(\theta) &= \hat{C}(\theta + \delta\theta) - \hat{C}(\theta) \\
&= \delta\theta^{\mu\nu} \frac{\partial \hat{C}}{\partial \theta^{\mu\nu}} = -\frac{1}{4}\theta^{\kappa\lambda}\{\hat{A}_\kappa, \partial_\lambda \hat{C}\}_*
\end{aligned}$$

These differential equations are commonly called SW differential equations.

To find solutions of the differential equation let us expand NC gauge parameter and NC gauge field into a Taylor series:

$$\begin{aligned}\hat{C}^{(n)} &= c + C^{(1)} + \cdots + C^{(n)},\\ \hat{A}_\mu^{(n)} &= A_\mu + A_\mu^{(1)} + \cdots + A_\mu^{(n)}.\end{aligned}$$

here $\hat{C}^{(n)}$ and $\hat{A}_\mu^{(n)}$ denotes the sum up to order n ! Then it is possible to write *(Wulkenhaar et.al'01)* :

$$\hat{C}_\alpha^{(n+1)} = \alpha - \frac{1}{4}\sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\cdots\theta^{\mu_k\nu_k}\left(\frac{\partial^{k-1}}{\partial\theta^{\mu_2\nu_2}\cdots\partial\theta^{\mu_k\nu_k}}\{\hat{A}_{\mu_1}^{(k)}, \partial_{\nu_1}\hat{C}_\alpha^{(k)}\}_*\right)_{\theta=0}$$

$$\hat{A}_\gamma^{(n+1)} = A_\gamma - \frac{1}{4}\sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\cdots\theta^{\mu_k\nu_k}\left(\frac{\partial^{k-1}}{\partial\theta^{\mu_2\nu_2}\cdots\partial\theta^{\mu_k\nu_k}}\{\hat{A}_{\mu_1}^{(k)}, \partial_{\nu_1}\hat{A}_\gamma^{(k)} + \hat{F}_{\nu_1\gamma}^{(k)}\}_*\right)_{\theta=0}$$

Let us write the $n+1$–st component of $\hat{C}_\alpha^{(n+1)}$:

$$C^{(n+1)} = -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\ldots\partial\theta^{\mu_n\nu_n}}\{\hat{A}_{\mu_1}^{(n)},\partial_{\nu_1}\hat{C}^{(n)}\}_*\right)_{\theta=0}.$$

Since, θ is set to zero after taking the derivatives, the expression in the parentheses can be written as a sum up to n-th order:

$$C^{(n+1)} = -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\ldots\partial\theta^{\mu_n\nu_n}}\sum_{p+q+r=n}\{A_\mu^{(p)},\partial_\nu C^{(q)}\}_{*^r}\right).$$

It is then an easy exercise to show that the above equation reduces to the recursive formula :

$$C^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\mu\nu}\sum_{p+q+r=n}\{A_\mu^{(p)},\partial_\nu C^{(q)}\}_{*^r}.$$

With the same algebraic manipulation one can also derive the same recursive formula for the gauge field

$$
\begin{aligned}
A_\gamma^{n+1} &= -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n} \times \\
&\qquad \times \left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\ldots\partial\theta^{\mu_n\nu_n}}\{\hat{A}_{\mu_1}^{(n)}, \partial_{\nu_1}\hat{A}_\gamma^{(n)} + \hat{F}_{\nu_1\gamma}^{(n)}\}_*\right)_{\theta=0} \\
&= -\frac{1}{4(n+1)}\theta^{\mu\nu}\sum_{p+q+r=n}\{A_\mu^p, \partial_\nu A_\gamma^q + F_{\nu\gamma}^q\}_{*^r}.
\end{aligned}
$$

Seiberg–Witten Map for Matter Fields

SW–map of a NC field $\hat{\Psi}$ in a gauge invariant theory can be derived from *(J. Wess et.al. EPJ'01)*:

$$\hat{s}\hat{\Psi}(\psi, A; \theta) = s\hat{\Psi}(\psi, A; \theta).$$

To find the solution of the gauge equivalence relation at each order in θ we expand the NC field $\hat{\Psi}$ as formal power series in θ

$$\hat{\Psi} = \psi + \Psi^{(1)} + \cdots + \Psi^{(n)} + \cdots$$

Fundamental Representation :

Ordinary BRST transformation of ψ is written as

$$s\psi = ic \cdot \psi$$

NC gauge transformation is defined with the help of $*$–product :

$$\hat{s}\hat{\Psi} = i\hat{C} * \hat{\Psi}.$$

Following the general strategy the gauge equivalence relation reads

$$\Delta\Psi^{(n)} \equiv s\Psi^{(n)} - ic \cdot \Psi^{(n)} = i \sum_{\substack{p+q+r=n, \\ q \neq n}} C^{(p)} *^r \Psi^{(q)},$$

for all orders.

As discussed before, one is free to add any homogeneous solution $\tilde{\Psi}^n$ of the equation

$$\Delta_\alpha \tilde{\Psi}^n = 0$$

to the solutions Ψ^n.

A solution for the first order is given by Wess et.al :

$$\Psi^{(1)} = -\frac{1}{4}\theta^{\kappa\lambda}A_\kappa(\partial_\lambda + D_\lambda)\psi$$

where $D_\mu\psi = \partial_\mu\psi - iA_\mu\psi$.

The SW differential equation from the first order solution reads [K.U, Yapiskan PRD'09]:

$$\delta\theta^{\mu\nu}\frac{\partial\hat{\Psi}}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\kappa\lambda}\hat{A}_\kappa * (\partial_\lambda\hat{\Psi} + \hat{D}_\lambda\hat{\Psi})$$

which can also be written as

$$\frac{\partial\hat{\Psi}}{\partial\theta^{\kappa\lambda}} = -\frac{1}{8}\hat{A}_\kappa * (\partial_\lambda\hat{\Psi} + \hat{D}_\lambda\hat{\Psi}) + \frac{1}{8}\hat{A}_\lambda * (\partial_\kappa\hat{\Psi} + \hat{D}_\kappa\hat{\Psi})$$

The NC covariant derivative here is defined as :

$$\hat{D}_\mu\hat{\Psi} = \partial_\mu\hat{\Psi} - i\hat{A}_\mu * \hat{\Psi} .$$

By using a similar method, we expand $\hat{\Psi}$ in Taylor series :

$$\begin{aligned} \hat{\Psi}^{(n+1)} &= \psi + \Psi^1 + \Psi^2 + \cdots + \Psi^{n+1} \\ &= \psi + \sum_{k=1}^{n+1} \frac{1}{k!} \theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2} \ldots \theta^{\mu_k\nu_k} \left(\frac{\partial^k}{\partial\theta^{\mu_1\nu_1} \ldots \partial\theta^{\mu_k\nu_k}} \left(\hat{\Psi}^{(n+1)}\right)\right)_{\theta=0}. \end{aligned}$$

From the differential equation we get

$$\begin{aligned} \hat{\Psi}^{(n+1)} &= \psi - \frac{1}{4}\sum_{k=1}^{n+1} \frac{1}{k!} \theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2} \ldots \theta^{\mu_k\nu_k} \times \\ &\quad \times \Big(\frac{\partial^{k-1}}{\partial\theta^{\mu_2\nu_2} \ldots \partial\theta^{\mu_k\nu_k}} \hat{A}^{(k)}_{\mu_1} * (\partial_{\nu_1}\hat{\Psi}^{(k)} + (\hat{D}_{\nu_1}\hat{\Psi})^{(k)})\Big)_{\theta=0} \end{aligned}$$

where

$$(\hat{D}_\mu \hat{\Psi})^{(n)} = \partial_\mu \hat{\Psi}^{(n)} - i\hat{A}^{(n)}_\mu * \hat{\Psi}^{(n)}.$$

For the abelian case this solution differs from the one given in Wulkenhaar et.al. by a homogeneous solution.

To find the all order recursive solution we write the $n+1$–st component :

$$\begin{aligned}
\Psi^{n+1} &= -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}\times \\
&\quad \times\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\dots\partial\theta^{\mu_n\nu_n}}\hat{A}^{(n)}_\mu * (\partial_\nu\hat{\Psi}^{(n)} + (\hat{D}_\nu\hat{\Psi})^{(n)})\right)_{\theta=0} \\
= \quad & -\frac{1}{4(n+1)!}\theta^{\mu\nu}\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}\times \\
& \times\left(\frac{\partial^n}{\partial\theta^{\mu_1\nu_1}\dots\partial\theta^{\mu_n\nu_n}}\sum_{p+q+r=n} A^p_\mu *^r(\partial_\nu\Psi^{(q)} + (D_\nu\Psi)^q)\right)
\end{aligned}$$

where

$$(D_\mu\Psi)^n = \partial\Psi^n - i\sum_{p+q+r=n} A^p_\mu *^r \Psi^q.$$

After taking derivatives w.r.t. θ's we obtain the all order recursive solution of the gauge equivalence relation :

$$\Psi^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} A_\kappa^{(p)} *^r (\partial_\lambda \Psi^{(q)} + (D_\lambda \Psi)^{(q)}).$$

The first order solution of Wess et.al. is obtained by setting $n=0$. By setting $n=1$ we find the second order solution of Möller .

$$\begin{aligned}\psi^{(2)} = (1/32)\theta^{\mu\nu}\theta^{\kappa\lambda}\big(& -4i\partial_\mu A_\kappa \partial_\lambda \partial_\nu \psi + 4A_\mu A_\kappa \partial_\lambda \partial_\nu \psi - 4\partial_\mu A_\kappa A_\nu \partial_\lambda \psi - 4A_\mu \partial_\kappa A_\nu \partial_\lambda \psi \\ & +8A_\mu \partial_\nu A_\kappa \partial_\lambda \psi - 2\partial_\mu A_\kappa \partial_\nu A_\lambda \psi + 4A_\mu A_\kappa A_\nu \psi - 3A_\mu A_\nu A_\kappa A_\lambda \psi - 2A_\mu A_\kappa A_\lambda A_\nu \psi + 4iA_\mu A_\kappa A_\nu \partial_\lambda \psi \\ & -4iA_\mu A_\kappa A_\lambda \partial_\nu \psi - 4iA_\mu A_\nu A_\kappa \partial_\lambda \psi + 2i\partial_\mu A_\kappa A_\nu A_\lambda \psi - 2iA_\mu A_\kappa \partial_\lambda A_\nu \psi - i\partial_\mu A_\kappa A_\lambda A_\nu \psi - 5iA_\mu \partial_\nu A_\kappa A_\lambda \psi \\ & +3iA_\mu \partial_\kappa A_\nu A_\lambda \psi - iA_\mu A_\kappa \partial_\nu A_\lambda \psi\big).\end{aligned}$$

Adjoint representation :
The BRST transformation reads as

$$s\psi = i[c, \psi]$$

Non–commutative generalization can be written as

$$\hat{s}\hat{\Psi} = i[\hat{C}_\alpha, \hat{\Psi}]_*$$

Following the general strategy, the gauge equivalence relation can be written as

$$\Delta\Psi^{(n)} \equiv \delta_\alpha \Psi^{(n)} - i[c, \Psi^{(n)}] = i \sum_{\substack{p+q+r=n, \\ q\neq n}} [\Lambda_\alpha^{(p)}, \Psi^{(q)}]_{*^r}$$

for all orders. The solutions can be found

- either by directly solving this equation order by order
- by solving the respective differential equation
- However, possibly the easiest way to obtain the solution is to use dimensional reduction. (K.U, Saka PRD'07)

By setting the components of the deformation parameter θ on the compactified dimensions zero, i.e.

$$\Theta^{MN} = \begin{pmatrix} \theta^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix},$$

the dimensional reduction (i.e. from six to four dimensions) of

$$A_N^{(n+1)} = -\frac{1}{4(n+1)}\theta^{KL} \sum_{p+q+r=n} \{A_K^{(p)}, \partial_L A_N^{(q)} + F_{LN}^{(q)}\}_{*^r}.$$

leads to the general n–th order solution for a complex scalar field :

$$\psi^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda} \sum_{p+q+r=n} \{A_\kappa^{(p)}, (\partial_\lambda \Psi^{(q)} + (D_\lambda \Psi)^{(q)})\}_{*^r}.$$

Since the structure of the solutions are the same for both the scalar and fermionic fields, this solution can also be used for the fermionic fields.

Note that, after introducing the anticommutators / commutators properly, the form of the solution is similar with the one given for the fundamental case except that now

$$D_\mu \psi = \partial_\mu \psi - i[A_\mu, \psi]$$

and hence

$$(D_\mu \Psi)^n = \partial_\mu \Psi^n - i \sum_{p+q+r=n} [A_\mu^p, \Psi^q]_{*^r} .$$

The same result can also be obtained by solving the differential equation :

$$\frac{\partial \hat{\Psi}}{\partial \theta^{\kappa\lambda}} = -\frac{1}{8}\{\hat{A}_\kappa, (\partial_\lambda \hat{\Psi} + \hat{D}_\lambda \hat{\Psi})\}_* + \frac{1}{8}\{\hat{A}_\lambda, (\partial_\kappa \hat{\Psi} + \hat{D}_\kappa \hat{\Psi})\}_* .$$

Therefore, the result obtained above via dimensional reduction can also be thought as an independent check of the aforementioned results.

Homogeneous Solutions :

Note the operator

$$\Delta \cdot = s \cdot - i\{c, \cdot]$$

commutes with the covariant derivative :

$$[\Delta, D_\mu] = 0 \Rightarrow \Delta F_{\mu\nu} = 0$$

Therefore at each order the form of the homogeneous solutions are obtained as

$$\tilde{A}_\gamma^{(n)} \propto \mathcal{F}_\gamma^{(n)}(\theta, D, F) \quad , \quad \tilde{\Psi}^{(n)} \propto \mathcal{P}^{(n)}(\theta, D, F)\psi$$

For instance 1st order homogeneous solutions :

$$\tilde{A}^{(1)}_{\gamma} \propto \theta^{\mu\nu} D_{\gamma} F_{\mu\nu} \quad , \quad \tilde{\Psi}^{(1)} \propto \theta^{\mu\nu} F_{\mu\nu} \psi$$

2nd order solutions :

$$\begin{aligned}\tilde{A}^{(2)}_{\gamma} \propto \quad & \theta^{\mu\nu}\theta^{\kappa\lambda} D_{\gamma}(F_{\mu\nu}F_{\kappa\lambda})\,,\ \theta^{\mu\nu}\theta^{\kappa\lambda} D_{\gamma}(F_{\mu\kappa}F_{\nu\lambda})\,,\ \theta^{\mu\nu}\theta^{\kappa\lambda} D_{\mu}(F_{\gamma\nu}F_{\kappa\lambda})\,, \\ & \theta^{\mu\nu}\theta^{\kappa\lambda} D_{\kappa}(F_{\mu\nu}F_{\gamma\lambda})\,,\ \theta^{\mu\nu}\theta^{\kappa\lambda} D_{\mu}(F_{\kappa\nu}F_{\gamma\lambda})\,,\ \theta^{\mu\nu}\theta^{\kappa\lambda} D_{\kappa}(F_{\mu\lambda}F_{\gamma\nu})\end{aligned}$$

$$\begin{aligned}\tilde{\Psi}^{(2)}_{\gamma} \propto \quad & \theta^{\mu\nu}\theta^{\kappa\lambda}(F_{\mu\nu}F_{\kappa\lambda})\psi\,,\ \theta^{\mu\nu}\theta^{\kappa\lambda}(F_{\mu\kappa}F_{\nu\lambda})\psi, \\ & i\theta^{\mu\nu}\theta^{\kappa\lambda}(D_{\mu}F_{\kappa\lambda})D_{\nu}\psi\,,\ i\theta^{\mu\nu}\theta^{\kappa\lambda}(D_{\mu}F_{\kappa\nu})D_{\lambda}\psi\end{aligned}$$

Contribution of the first order homogeneous solutions to the second order:

It is clear from the equations that define the all order solutions

$$\Delta A_\mu^{(n)} \equiv s A_\mu^{(n)} - i[c, A_\mu^{(n)}] \;=\; \partial_\mu C_\alpha^{(n)} - i \sum_{\substack{p+q+r=n,\\ p\neq n}} [A_\mu^{(p)}\,,\, C_\alpha^{(q)}]_{*^r}$$

$$\Delta \Psi^{(n)} \equiv s\Psi^{(n)} - ic\cdot\Psi^{(n)} \;=\; i \sum_{\substack{p+q+r=n,\\ q\neq n}} C^{(p)} {*^r} \Psi^{(q)}$$

the homogeneous solutions of lower orders will contribute to the higher order ones.

In order to find these contributions for the second order let us decompose the fields at the first order fields as

$$A^{(1)} \to A^{(1)} + \tilde{A}^{(1)} \quad , \quad \Psi^{(1)} \to \Psi^{(1)} + \tilde{\Psi}^{(1)}$$
$$A^{(2)} \to A^{(2)} + \bar{A}^{(2)} + \tilde{A}^{(2)} \quad , \quad \Psi^{(2)} \to \Psi^{(2)} + \bar{\Psi}^{(2)} + \tilde{\Psi}^{(2)}$$

Since $\Delta\tilde{A}^{(2)} = \Delta\tilde{\Psi}^{(2)} = 0$ we get

$$\Delta\bar{A}^{(2)}_{\gamma} = i[C^{(1)}, \tilde{A}^{(1)}_{\gamma}] - \frac{1}{2}\theta^{\kappa\lambda}\{\partial_\kappa c, \partial_\lambda \tilde{A}^{(1)}_{\gamma}\}$$
$$\Delta\bar{\Psi}^{(2)} = iC^{(1)} \cdot \tilde{\Psi}^{(1)} - \frac{1}{2}\theta^{\kappa\lambda}\partial_\kappa c \cdot \partial_\lambda \tilde{\Psi}^{(1)}$$

The solution of these equations are given as

$$\bar{A}^{(2)}_{\gamma} = -\frac{1}{4}\theta^{\kappa\lambda}(2\{A_\kappa, \partial_\lambda \tilde{A}^{(1)}_{\gamma}\} - i\{A_\kappa, [A_\lambda, \tilde{A}^{(1)}_{\gamma}]\})$$
$$\bar{\Psi}^{(2)} = -\frac{1}{4}\theta^{\kappa\lambda}A_k(2\partial_\lambda \tilde{\Psi}^{(1)} - iA_\lambda \cdot \tilde{\Psi}^{(1)})$$

CONCLUSION

Equations,

$$\hat{A}_\mu(A;\theta) + \hat{\delta}_{\hat{\Lambda}}\hat{A}_\mu(A;\theta) = \hat{A}_\mu(A + \delta_\alpha A;\theta).$$

$$\hat{\Psi}(A,\psi;\theta) + \hat{\delta}_{\hat{\Lambda}}\hat{\Psi}(A,\psi;\theta) = \hat{\Psi}(A + \delta_\alpha A, \psi + \delta_\alpha \psi;\theta).$$

are solved by

$$\Lambda_\alpha^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}\{A_\kappa^p, \partial_\lambda \Lambda_\alpha^q\}_{*^r}$$

$$A_\gamma^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}\{A_\kappa^p, \partial_\lambda A_\gamma^q + F_{\lambda\gamma}^q\}_{*^r}.$$

$$\psi^{n+1} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\sum_{p+q+r=n}A_\kappa^p {*^r}(\partial_\lambda \Psi^{(q)} + (D_\lambda \Psi)^q).$$